PHYSICS NYB-10/11 Winter 2007

Lecture 14: Continuous charge distributions: electric field

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Review

- The potential associated with a single point charge q is
$$V(r) = k_e \frac{q}{r}$$
- If we consider the potential due to multiple charges Δq_i, we take the sum of the potentials from each charge
$$V_{tot}(r) = \sum_i k_e \frac{\Delta q_i}{r_i}$$
- For a continuous charge distribution this sum becomes an integral.
$$V_{tot}(r) = \int k_e \frac{dq}{r}$$

Continuous charge distributions

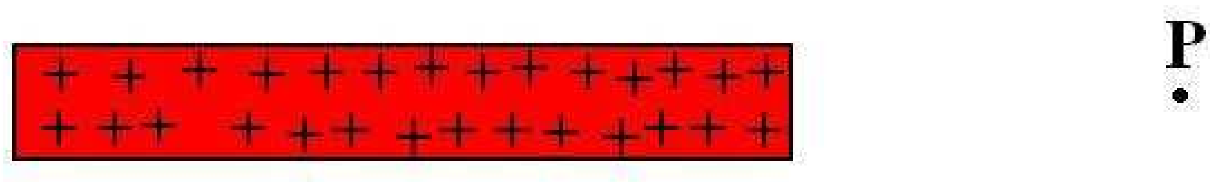
In the previous lecture, we managed to find the electric potential at point P due to a uniformly charged rod. Now, we'd like to find the electric *field* at point P.

Continuous charge distributions

Just like for the potential, we notice that the rod is not a point charge...

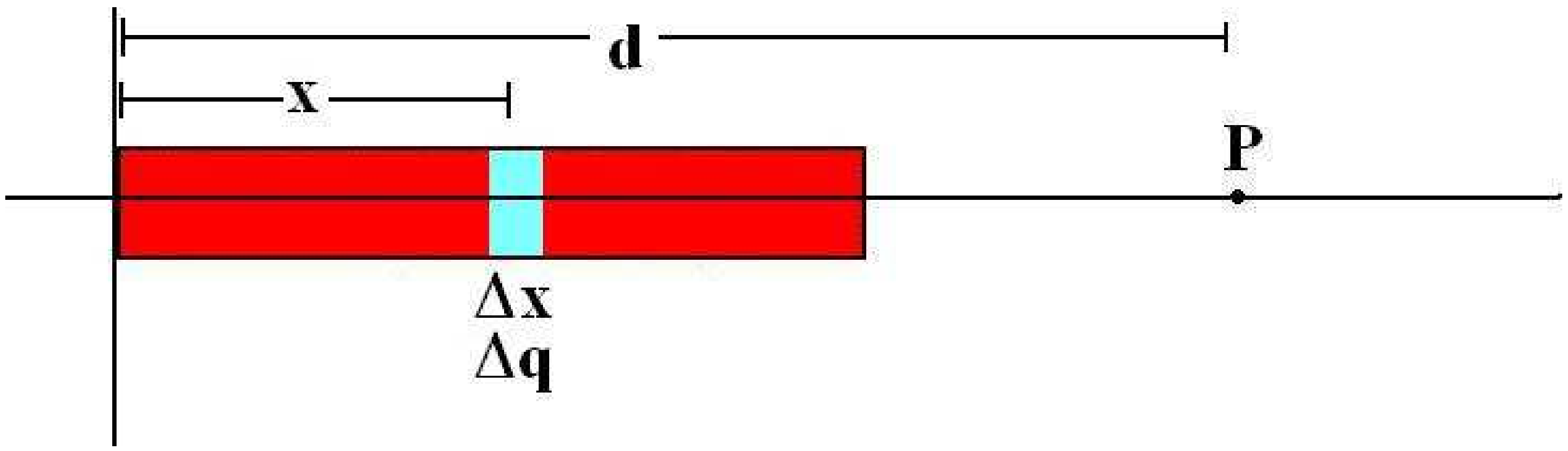
However, a plastic rod *is made of a whole bunch of points*.

We *do* know how to find the field for each of these points. And we *do* know how to add vectors...



Continuous charge distributions

So let's look at the electric field from a small part of the rod, of length Δx and carrying an amount of charge Δq. If we take a small enough piece, it is perfectly ok to treat it as a point.



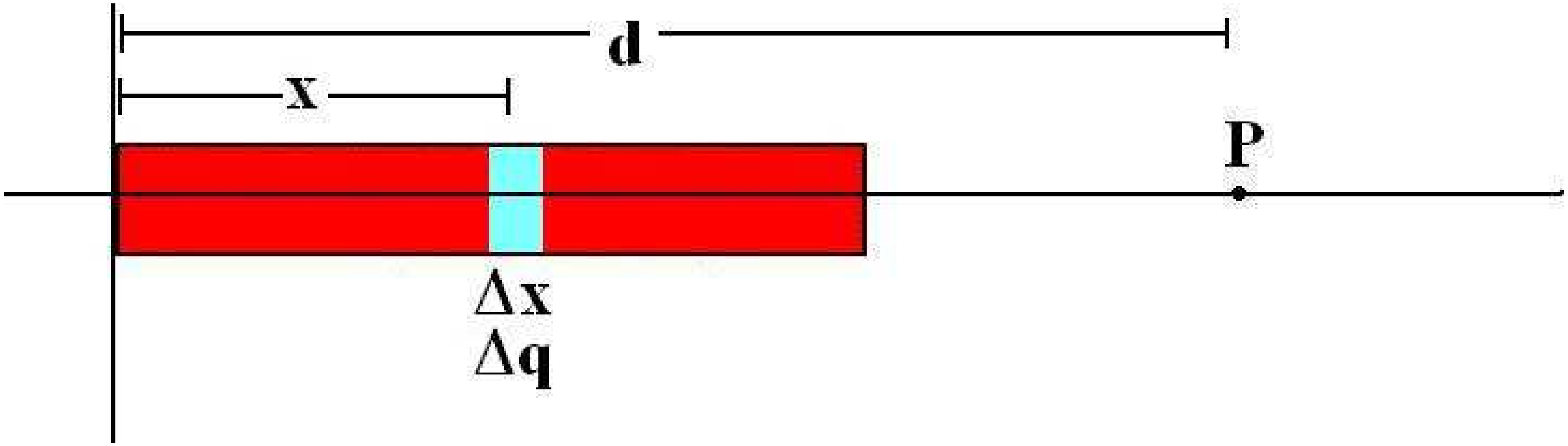
The small field due to this small piece of the rod is

$$\Delta \vec{E} = k_e \frac{\Delta q}{r^2} \hat{i} .$$

But how much charge is on the piece? What is Δq equal to?

Continuous charge distributions

Using what we learned last time concerning charge density, we know that the amount of charge on our piece of length Δx is $\Delta q = \lambda \Delta x$.



$$\Delta \vec{E} = k_e \frac{\lambda \Delta x}{r^2} \hat{i}$$

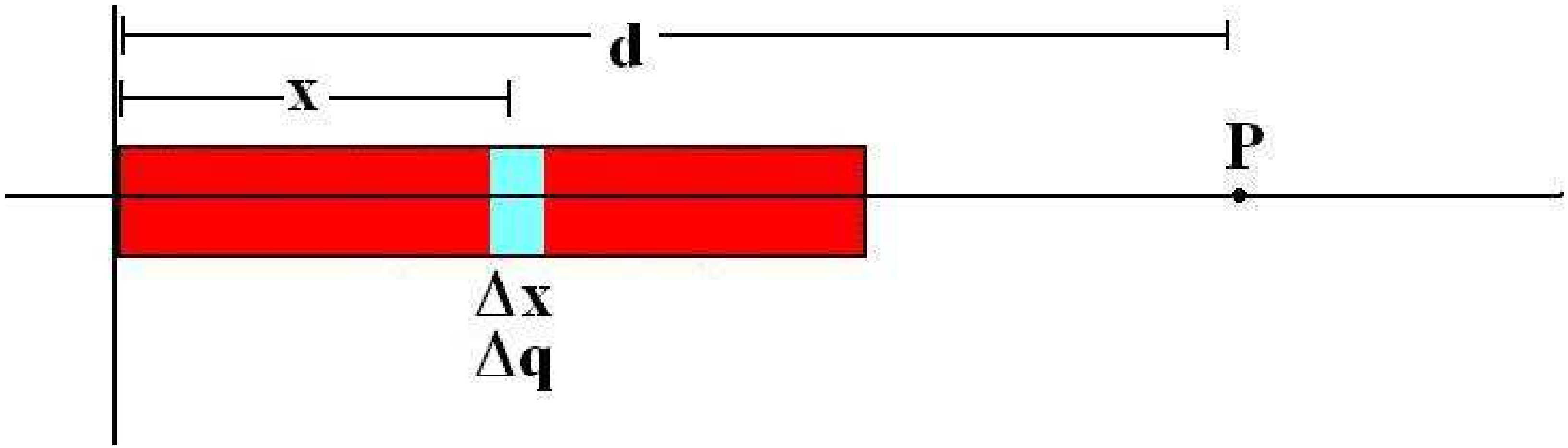
.

Continuous charge distributions

To get the total field due to the entire rod, we need to add the potential due to every small piece. So

$$\vec{E}_{tot} = \Delta\vec{E}_1 + \Delta\vec{E}_2 + ... \equiv \sum_i \Delta\vec{E}_i.$$

Note however that the distance r from the point P to each small piece is different. We can write $r_i = d - x_i$



$$\vec{E}_{tot} = \sum_i \Delta\vec{E}_i = \sum_i k_e \frac{\lambda \Delta x_i}{(d - x_i)^2}\hat{i}$$

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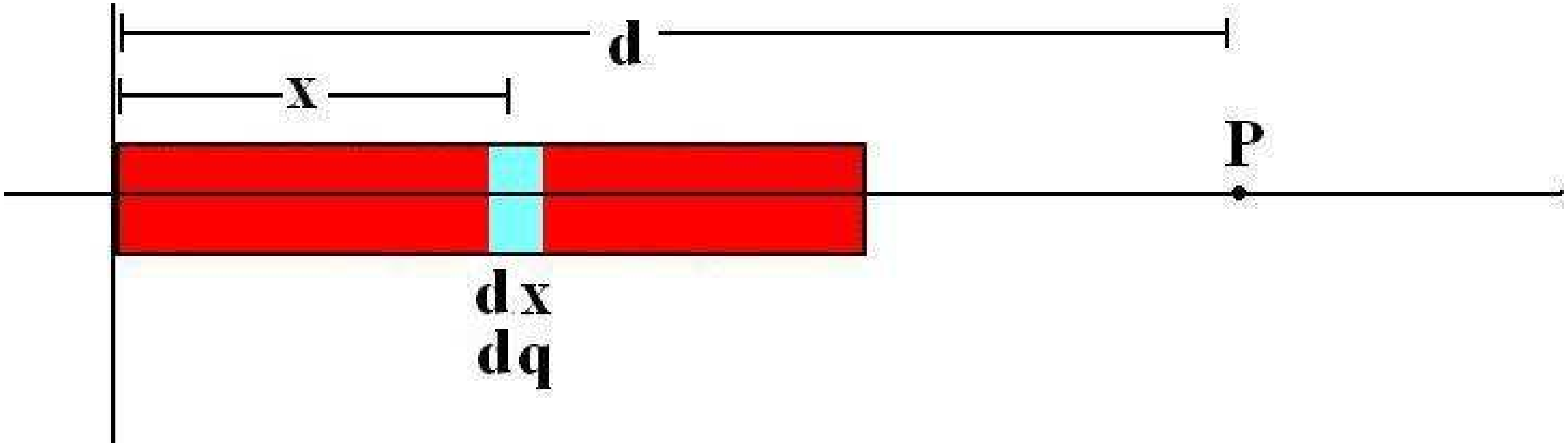
Continuous charge distributions

The last step is to make each piece infinitely small, so that $\Delta x_i \rightarrow dx$ and the sum becomes an infinite sum of infinitely small pieces, in other words *an integral*.



$$\vec{E}_{tot} = \sum_i \Delta\vec{E}_i = \sum_i k_e \frac{\lambda \Delta x_i}{(d - x_i)^2}\hat{i} \rightarrow \vec{E}_{tot} = \int d\vec{E} = \int_0^L k_e \frac{\lambda dx}{(d - x)^2}\hat{i}$$

Continuous charge distributions

Now we evaluate the integral

$$\vec{E}_{tot} = \int_0^L k_e \lambda \frac{dx}{(d-x)^2}\hat{i} = k_e \lambda \int_0^L \frac{dx}{(d-x)^2}\hat{i}$$

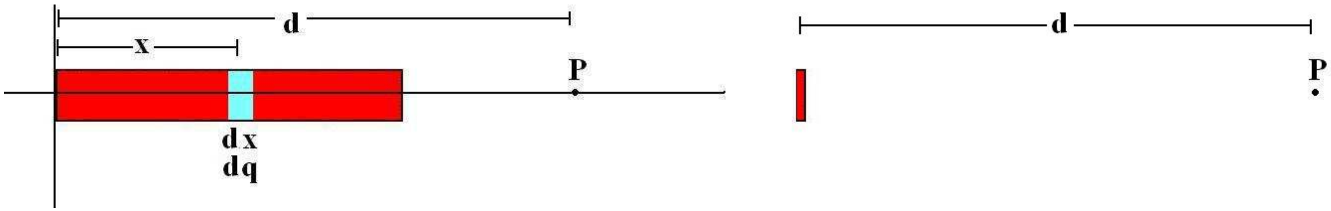
First we'll make a change of variables, $d - x = u$ which means $dx = -du$ and the boundaries of integration become from d to $d - L$. So

$$\begin{aligned}
\vec{E}_{tot} &= k_e \lambda \int_d^{d-L} \frac{-du}{u^2}\hat{i} = k_e \lambda \left[\frac{1}{u}\right]_d^{d-L} \hat{i} \\
&= k_e \lambda \left[\frac{1}{d-L} - \frac{1}{d}\right]\hat{i} = k_e \lambda \left[\frac{d-(d-L)}{d(d-L)}\right]\hat{i} \\
&= k_e \lambda \frac{L}{d(d-L)}\hat{i} = k_e \frac{Q}{d(d-L)}\hat{i}
\end{aligned}$$

Continuous charge distributions

$$\vec{E}_{tot} = k_e \frac{Q}{d(d-L)}$$

Let's check if this result makes sense, at least in some simplified situation. Imagine the rod was very very short, then we should get back the same result as for a point charge.



Clearly here, when $L \to 0$, we get $\vec{E} = k_e \frac{Q}{d^2}$, which is precisely what you'd expect.

$\vec{E}$ vs V

$$\vec{E}_{tot} = k_e \frac{Q}{d(d-L)}$$

Note also that d is actually the x-coordinate of point P. This means that we could write the field along any point on the x-axis as $\vec{E}_{tot} = k_e \frac{Q}{x(x-L)}$. Remember that last time, we found that the potential at point P was

$V_{tot} = k_e \frac{Q}{L} \ln\left(\frac{d}{d-L}\right)$ which we can write as

$V_{tot} = k_e \frac{Q}{L} \ln\left(\frac{x}{x-L}\right)$ for any point on the x-axis. Now we'd like to check that the relation $E_x = -\frac{\partial V}{\partial x}$.

$\vec{E}$ vs V

$$
\begin{aligned}
E_x = -\frac{\partial}{\partial x}V_{tot} &= -\frac{\partial}{\partial x}\left(k_e\frac{Q}{L}\ln\left(\frac{x}{x-L}\right)\right) \\
&= -k_e\frac{Q}{L}\frac{x-L}{x}\times\frac{\partial}{\partial x}\left(\frac{x}{x-L}\right) \\
&= -k_e\frac{Q}{L}\frac{x-L}{x}\left(\frac{1}{x-L}+\frac{-x}{(x-L)^2}\right) \\
&= -k_e\frac{Q}{L}\left(\frac{1}{x}-\frac{1}{x-L}\right) = -k_e\frac{Q}{L}\left(\frac{x-L-x}{x(x-L)}\right) \\
E_x &= -k_e\frac{Q}{L}\left(\frac{-L}{x(x-L)}\right) = k_e\frac{Q}{x(x-L)}
\end{aligned}
$$

which is exactly what we expected.

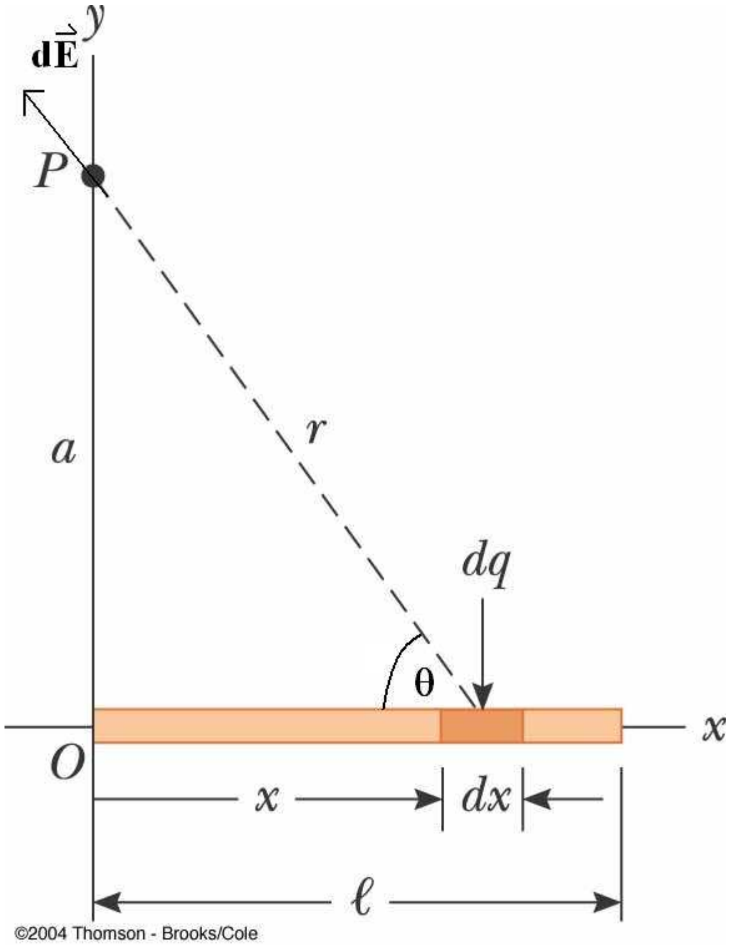
Continuous charge distributions

- When faced with a continuous charge distribution, use the following steps:
  - Choose your coordinates wisely.
  - Check if you can use *symmetry arguments* to simplify the problem.
  - Choose one small piece of the distribution containing a charge Δq.
  - Write down the electric field vector at the point P for this single piece.
  - If there are components of $\vec{E}$ in different directions, treat them separately from this point on.

Continuous charge distributions

- Note that so far, you haven't done anything harder than finding the electric field for a single point charge.
  - The key step comes next in writing Δq in terms of the charge *density* and the size of the small piece.
  - Make sure everything is written in terms of constants and the coordinates you chose.
  - Now take a sum of all the small pieces the distribution is made of by taking an integral; *make sure you correctly identify which coordinate(s) to integrate, and over what range.*
  - It's always a good idea to check that your answer reduces to something you know in an appropriate limit.

Examples



If this rod has a uniform linear charge density λ, find the electric field at point P.

Examples

First, let us find the magnitude of the electric field created at point P by the small piece of the rod shown on the previous slide. It will be given by

$$dE = k_e \frac{dq}{r^2} = k_e \frac{dq}{x^2 + a^2}$$

Now the piece of the rod we are looking at has a length dx and a charge density λ C/m, so that the charge on it is $dq = \lambda dx$. So

$$dE = k_e \frac{\lambda dx}{x^2 + a^2}$$

This electric field will be pointing directly away from the piece of the rod we're considering, and we must now break it down into its x and y components. From the diagram, we see that

$$d\vec{E} = dE_x \hat{i} + dE_y \hat{j} = -dE \cos\theta \hat{i} + dE \sin\theta \hat{j}$$

Examples

Notice that as we look at different pieces of the rod located at different values of x, the angle θ will change, so we have to write $\sin\theta$ and $\cos\theta$ in terms of x. From the diagram, we see that $\cos\theta = x/\sqrt{x^2+a^2}$ and $\sin\theta = a/\sqrt{x^2+a^2}$. So now we can write

$$d\vec{E} = -k_e \frac{\lambda x dx}{(x^2+a^2)^{3/2}}\hat{i} + k_e \frac{\lambda a dy}{(x^2+a^2)^{3/2}}\hat{j}$$

So now we'll have to perform one integral to find the total field in the x direction, and another integral to find the total field in the y direction.

Examples

$$
\begin{aligned}
E_x &= \int_0^L dE_x = -k_e\lambda \int_0^L \frac{x\,dx}{(x^2+a^2)^{3/2}} \\
u &\equiv x^2 + a^2; \qquad du = 2x\,dx \\
E_x &= -k_e\lambda \int \frac{du/2}{u^{3/2}} = -\frac{k_e\lambda}{2}\left[-\frac{2}{\sqrt{u}}\right] \\
&= k_e\lambda \left[\frac{1}{\sqrt{x^2+a^2}}\right]_0^L = k_e\lambda\left[\frac{1}{\sqrt{L^2+a^2}} - \frac{1}{a}\right] \\
&= -k_e\lambda\left[\frac{1}{a} - \frac{1}{\sqrt{L^2+a^2}}\right]
\end{aligned}
$$

Examples

$$
\begin{aligned}
E_y &= \int_0^L dE_y = k_e \lambda a \int_0^L \frac{dx}{(x^2+a^2)^{3/2}} \\
x &\equiv a \tan\alpha; \qquad dx = a d\alpha / \cos^2\alpha; \\
x^2 + a^2 &= a^2(1+\tan^2\alpha) = a^2/\cos^2\alpha \\
E_y &= k_e \lambda a \int \frac{a d\alpha}{\cos^2\alpha} \frac{\cos^3\alpha}{a^3} = k_e \frac{\lambda}{a} \int \cos\alpha d\alpha \\
&= k_e \frac{\lambda}{a} [\sin\alpha]
\end{aligned}
$$

Examples

Now we need to write $\sin\alpha$ in terms of x. We have

$$
\begin{aligned}
x &= a\tan\alpha \\
1+\frac{x^2}{a^2} &= 1+\tan^2\alpha = \frac{1}{\cos^2\alpha} = \frac{1}{1-\sin^2\alpha} \\
1-\sin^2\alpha &= \frac{1}{1+x^2/a^2} \\
\sin^2\alpha &= 1-\frac{1}{1+x^2/a^2} = \frac{1+x^2/a^2-1}{1+x^2/a^2} = \frac{x^2}{x^2+a^2} \\
\sin\alpha &= \frac{x}{\sqrt{x^2+a^2}}
\end{aligned}
$$

Examples

Now we have

$$E_y = k_e \frac{\lambda}{a}\left[\frac{x}{\sqrt{x^2+a^2}}\right]_0^L$$

$$E_y = k_e \frac{\lambda L}{a\sqrt{L^2+a^2}}$$

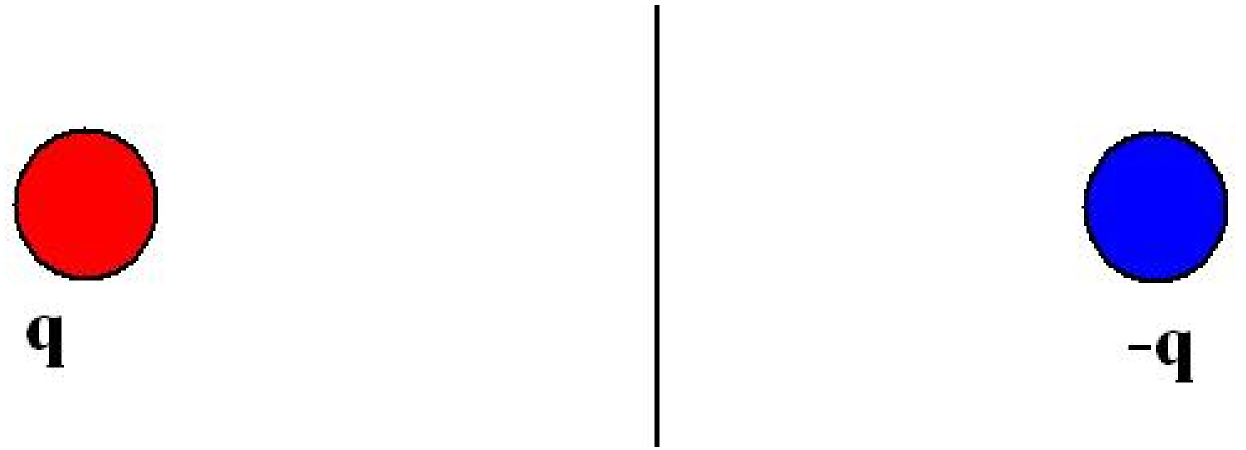
So finally,

$$\vec{E} = -k_e\lambda\left[\frac{1}{a} - \frac{1}{\sqrt{L^2+a^2}}\right]\hat{i} + k_e\frac{\lambda L}{a\sqrt{L^2+a^2}}\hat{j}$$

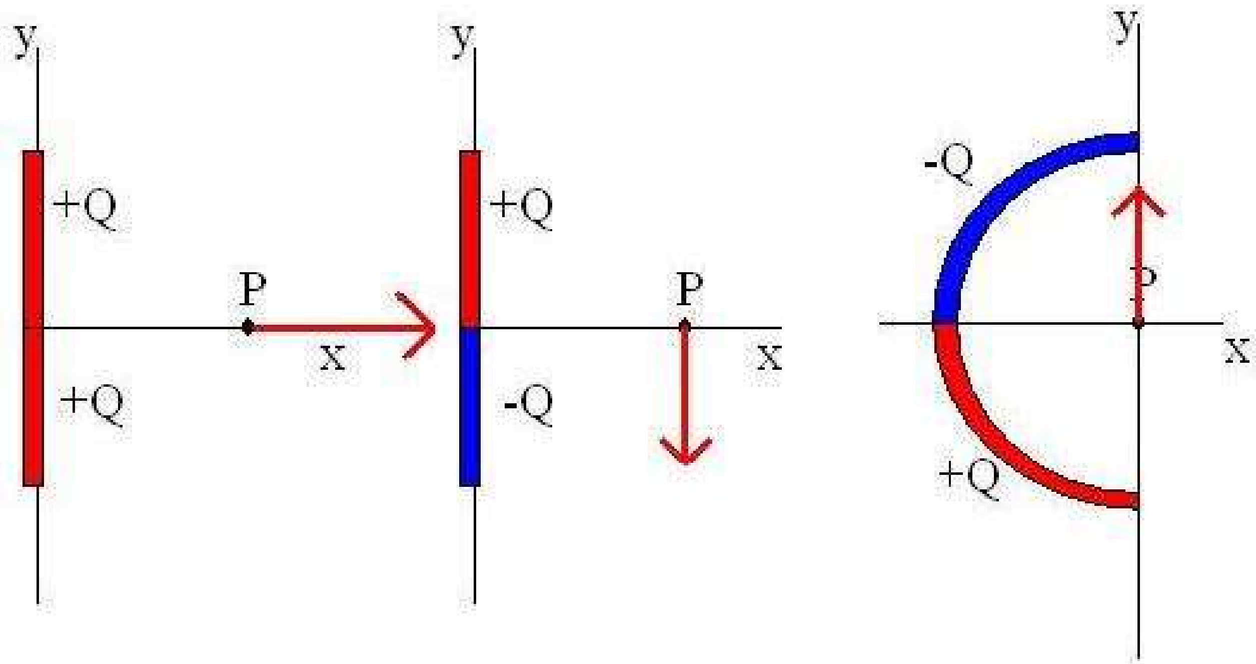
Using symmetry

Many times, you are able to gain very important information about the potential or the electric field simply by looking at the way the source charges are arranged.

For example, in the example below, you should be able to tell that the electric field everywhere along the line is pointing directly to the right without using any math...



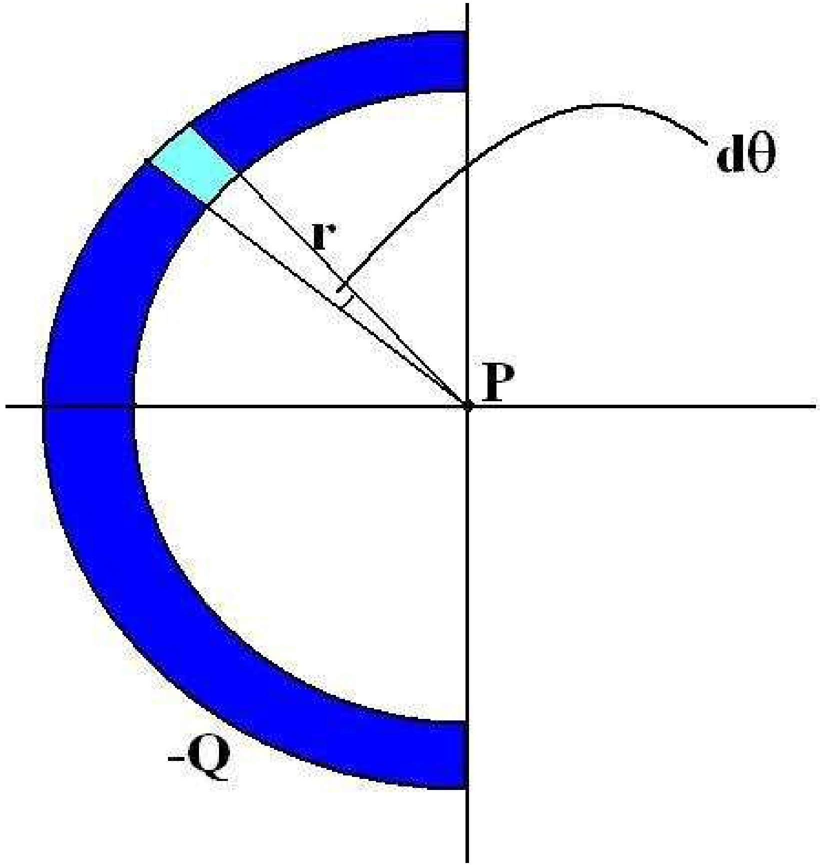
Examples



Question: The figure above shows three non conducting rods. Each has a uniform charge Q along the top and bottom halves. For each rod, what is the direction of the electric field at point P?

Examples

What is the electric field at point P at the center of a semi-circular arc of uniformly distributed charge $-Q$ and radius r?



Examples

First off, every small piece of the negatively charged arc creates an electric field pointing towards the arc. For every piece on the bottom half that has a downward y component of the electric field, we have a piece on the upper half that has an equal upward y component of the electric field, so that the net electric field will have to be only in the x direction. Therefore, we only need to perform the integral for the x component of the electric field.

Now let's find the magnitude of the electric field due to a small piece of the arc

$$dE = k_e \frac{dq}{r^2} = k_e \frac{\lambda ds}{r^2}$$

and the arc length ds is equal to $r d\theta$.

Examples

The x component of the electric field due to this small piece of arc is

$$dE_x = -dE\cos\theta = -k_e \frac{\lambda r d\theta}{r^2}\cos\theta$$

We now are ready to integrate this, noting that r is a constant, and the angle θ goes from $-\pi/2$ to $\pi/2$.

$$\begin{aligned}
E_x &= -\int dE_x = k_e \frac{\lambda}{r}\int_{-\pi/2}^{\pi/2}\cos\theta d\theta \\
&= -k_e \frac{\lambda}{r}\left[\sin\theta\right]_{-\pi/2}^{\pi/2} = -k_e\frac{\lambda}{r}[1-(-1)] \\
&= -2k_e\frac{\lambda}{r}
\end{aligned}$$

So $\vec{E} = -2k_e\frac{\lambda}{r}\hat{i} = -2k_e\frac{Q}{\pi r^2}\hat{i}$

What to read for next lecture

- Next lecture is a tutorial, where you'll be working on your integrative activity assignment.
- After that, we'll be starting Gauss's law. Read sections 23.6, 24.1 and 24.2.